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ABSTRACT

The usefulness of path analysis as a means of better understanding various linear models is demonstrated. First, two linear models are presented in matrix form using linear structural relations (LISREL) notation. The two models, regression and factor analysis, are shown to be identical although the research question and data matrix to which these models are applied are different. The second section demonstrates how path analysis may be used to enhance understanding of these models; path analysis more clearly reveals their similarity than expectation algebra. Finally, path analysis is applied in complex LISREL models, with an example of matrices rather than scalars. A two-wave multitrait, multiple indicator longitudinal LISREL model is used. (GDC)

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USE OF PATH ANALYSIS AND PATH DIAGRAMS AS A MEANS OF UNDERSTANDING REGRESSION, FACTOR ANALYSIS, AND OTHER LINEAR STRUCTURAL RELATIONS (LISREL) MODELS

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INTRODUCTION

The path analysis of causal models was developed by the geneticist Sewall Wright over 60 years ago (Wright, 1921) and in the 1960's-70's was introduced to social scientists by Duncan (1966, 1975), Blalock (1971), and others. During this time, almost independently, various causal models were developed in econometrics called simultaneous equation models (e.g., see Wonnacott and Wonnacott, 1970). These models were usually treated algebraically rather than with the graphic methods of path analysis.

One of the major problems in all the causal models was the issue of parameter estimation. This was especially true for cases in which reciprocal causality, or measurement error in the exogenous variables, was assumed. However, during the 1960's-70's these estimation problems and many others were solved by Karl Jöreskog (1970). Not only did Jöreskog solve many of the statistical problems associated with causal modeling, but he integrated many of the seemingly different models by showing how they are all related to one overall model. Thus, Jöreskog brought about an Einsteinian Revolution in statistics. Jöreskog's general LISREL model subsumes, as special cases, almost every conceivable linear model.

Along with the popularization of Jöreskog's clever matrix formulations it is important not to forget about the benefits of a path analytic rendering of linear models. This is true, not only for the researcher testing a model, but also for the teacher explaining linear models. Although Jöreskog's models are fascinating to deal with it is sometimes easy not to see the forest for the trees. One of the advantages of path analysis is that it uses an easily understandable graphic approach to achieve the same results as expectation algebra. It has the advantage of intimately acquainting the modeler with the model and as we will see, may be used to clearly see the similarities and differences between models. The purpose of this paper, then, is to demonstrate the usefulness of path analysis as a means of better "understanding" various linear models.

The paper will do this in three parts. In part I, two linear models will be presented in matrix form using LISREL notation. These will be models for regression and factor analysis. In this section, it will be stressed that these models are really identical although the research question and data matrix to which these models are applied are different. In part II, it will be demonstrated how path analysis may be used to enhance our understanding of these models. It is shown that the path analysis of these models more clearly reveals their similarity than expectation algebra. In part III, it will be

shown how path analysis may be used in complex LISREL models. This will be done with an example of the path analysis of matrices rather than scalars. The example that will be used is a two-wave multitrait, multiple indicator longitudinal LISREL model. It will be demonstrated that even for complicated LISREL systems, path analysis helps the researcher to simplify and gain insight into the model. This section of the paper will assume the reader is aware of the notation system of the general LISREL model.

Since the purpose of this paper is to enhance understanding of linear models a number of issues which contribute to confusion will be ignored. This paper will ignore issues of parameter estimation (i.e., we will assume all parameters are known), model specification, identification, and hypothesis testing. Furthermore, it will be assumed that all observed variables are measured with equal intervals, have been standardized, and exhibit no interactions.

PART I: MATRIX PRESENTATION OF REGRESSION AND FACTOR ANALYSIS MODELS

In univariate regression models it is specified that the relationship between a set of dependent variables Y and a set of independent variables X is of the form

$$Y=\Gamma X+\zeta \qquad (1)$$

In equation 1, Y is a p×1 vector of dependent variables, Γ is a p×q matrix of regression coefficients, X is a q×1 vector of independent variables, and ζ is a p×1 vector of residual variables. The assumptions needed to estimate the parameters of this model are

$$E(X\zeta')=0, \text{ and}$$
$$E(\zeta\zeta')=\Psi \text{ diagonal},$$

and we define the variance-covariance matrix of the independent variables as

$$E(XX')=\Phi .$$

With these assumptions and definitions the overall (p×q)×(p×q) variance-covariance super matrix S of the observed dependent and independent variables can be derived through expectation.

$$S=\left[\begin{array}{c|c} E(YY') & E(YX') \\ \hline E(XY') & E(XX') \end{array}\right]$$

$$=\left[\begin{array}{c|c} E[(\Gamma X+\zeta)(\Gamma X+\zeta)'] & E[(\Gamma X+\zeta)(X')] \\ \hline E[(X)(\Gamma X+\zeta)'] & E(XX') \end{array}\right]$$

$$=\left[\begin{array}{c|c} \Gamma\Phi\Gamma'+\psi & \Gamma\Phi \\ \hline \Phi\Gamma' & \Phi \end{array}\right] \qquad (2)$$

Equation 2 contains many familiar and useful deductions from the linear regression model. The upper left matrix shows that the variance-covariance of the dependent variables may be partitioned into additive components due to regression ($\Gamma\Phi\Gamma'$) and residual (ψ). The upper right matrix in equation 2 (and its transpose in the bottom left) is the set of normal equations which, when solved, yields parameter estimates of Γ. The matrix $\Gamma\Phi$ represents p sets of q normal equations, or one set of q normal equations for each dependent variable. Next, we will see that the linear regression model in equation 1 and its variance-covariance structure in equation 2 each have their counterparts in factor analysis.

In common factor analysis models a linear model is specified which indicates the relationship between a set of observed dependent variables and a set of unobserved independent variables (factors). The form of the relationship is

$$Y=\Lambda_y\eta+\varepsilon \qquad (3)$$

In equation 3, Y is a p×1 vector of observed dependent variables, Λ_y is a p×m factor pattern matrix (or a matrix of regression coefficients), η is an m×1 vector of unobserved factors, and ε is a p×1 vector of unique error variables. For estimation purposes, we assume

$$E(\eta\varepsilon') = 0, \text{ and}$$

$$E(\varepsilon\varepsilon') = \theta_\varepsilon \text{ diagonal.}$$

Furthermore, we define the variance-covariance matrix of the first order factors η as

$$E(\eta\eta') = \Omega_{\eta\eta}$$

With these assumptions and definitions the overall (p×m)×(p×m) variance-covariance supermatrix S of the observed dependent and underlying factors can be derived

$$S = \left[\begin{array}{c|c} E(YY') & E(Y\eta') \\ \hline E(\eta Y') & E(\eta\eta') \end{array}\right]$$

$$= \left[\begin{array}{c|c} E[(\Lambda_y\eta+\varepsilon)(\Lambda_y\eta+\varepsilon)'] & E[(\Lambda_y\eta+\varepsilon)(\eta')] \\ \hline E[(\eta)(\Lambda_y\eta+\varepsilon)'] & E(\eta\eta') \end{array}\right]$$

$$= \left[\begin{array}{c|c} \Lambda_y\Omega_{\eta\eta}\Lambda_y'+\theta_\varepsilon & \Lambda_y\Omega_{\eta\eta} \\ \hline \Omega_{\eta\eta}\Lambda_y' & \Omega_{\eta\eta} \end{array}\right] \qquad (4)$$

Like equation 2 for regression, equation 4 contains many useful deductions from the factor analysis model. The upper

left matrix of equation 4 shows that the variance-covariance of the dependent measures may be partitioned into two additive components. The first is due to the regression of the observed variables on the factors $\Lambda_y \Omega_{\eta\eta} \Lambda_y'$, and the second a residual term θ_ε. The diagonal elements of $\Lambda_y \Omega_{\eta\eta} \Lambda_y'$ are called communalities. If $\Omega_{\eta\eta}$ is assumed to be diagonal, the model is an orthogonal, rather than an oblique factor model. On the other hand if θ_ε is assumed to be 0 the model is a principal component rather than a factor model.

In addition, the matrix $\Lambda_y \Omega_{\eta\eta}$ in the upper right portion of equation 4 (and its transpose in the lower left) is called the factor structure matrix. This matrix shows the covariance between the factors and the observed dependent variables. The factor structure matrix should not be confused with the factor pattern matrix Λ_y which shows the causal impact of each factor on each observed variable. The factor structure matrix is comparable to the normal equations matrix in regression models.

The above regression and factor analysis models have been presented as a background for Part II of this paper. Although various similarities between regression and factor analysis have been pointed out, the path analysis of these models will make these similarities more obvious.

PART II: PATH ANALYSIS OF REGRESSION AND FACTOR ANALYSIS MODELS

Path analysis is essentially a way of reading path diagrams for the purpose of analyzing the causal components of a model. The two primary goals are to decompose the variances and covariances associated with endogeneous variables. This can be done by reading path diagrams according to three simple rules.

Rule 1: No path may pass through a variable more than once.

Rule 2: A path may not pass through a double headed arrow more than once.

Rule 3: After a path has gone forward on an arrow, it may not then go backwards on an arrow.

To apply these rules, imagine you are trying to go from one variable to another taking as many paths as possible without violating Wright's rules. Each path represents a component of a variance or covariance associated with an endogeneous variable. Furthermore, the number of arrows followed during each path (along with variances or covariances of exogenous variables) represents the number of terms in each component.

To see how this works a path diagram of a regression model is presented in Figure 1. The model indicates we are regressing three dependent variables on two independent variables. In Figure 1, the single-headed arrow indicates a causal impact, the double-headed arrow indicates a covariance, and the box

around the variable indicates the variable is observed (rather than latent). The path diagram in Figure 1 not only provides a clear visual graph of a regression model, but also may be read using Wright's rules to obtain various decompositions. Three decompositions of interest are presented in Figures 2, 3, and 4. These are: R^2 in each dependent variable (Figure 2); the covariance between each dependent variable (Figure 3); and the covariance between each independent and each dependent variable (Figure 4). As an example of Wright's rules, look at the covariance between Y_1 and Y_2 at the top of Figure 3. We see that this covariance has four components; i.e., there are four different paths from Y_1 to Y_2. Similarly, there are four components to the covariance between Y_1 and Y_3 and Y_2 and Y_3.

In practice regression models are different from factor analysis models in terms of notation, identification, and parameter estimation procedures. However, the purpose of this exercise is to use path analysis to show how the underlying model characteristics of regression and factor analysis are identical. To do this a three dependent variable, two factor path diagram is presented in Figure 5. The only difference between this model and the regression model in Figure 1 is the notation and the fact that the factors are unobserved (indicated by the circle rather than a square around η_1 and η_2). Using the rules of path analysis, communalities are derived in

Figure 6, covariances between dependent variables in Figure 7, and the factor structure matrix in Figure 8. A visual inspection will reveal that the causal components of the regression model in Figures 2, 3, and 4 are really identical to those of the factor analysis model in Figures 6, 7, and 8, respectively.

PART III: THE PATH ANALYSIS OF LISREL MATRICES

One of the frequent criticisms of path analysis is that the method is not useful for complex models. This is true only if the path analysis deals with the individual elements of the LISREL matrices. However, path analysis at the matrix level not only can result in the same deductions obtained with matrix expectation, but can often provide additional insight into the causal activities of the model. This part of the paper will provide an example of the use of the path analysis of matrices as a way of better understanding complex LISREL models. The presentation assumes the reader is aware of the notation system of the general LISREL model. If not, a very readable presentation of it has been provided by Long (1976).

Figure 9 presents a two-wave multivariable multiple-indicator longitudinal LISREL model (Jöreskog, 1977). The notation in this path diagram is like that in Figures 1 and 5 in every

way, except each entry is a matrix rather than a scalar. For example, $\underline{\Lambda}_{y_{11}}$ is a matrix of path coefficients (i.e., an entire factor pattern matrix) rather than an individual path coefficient. The LISREL measurement model for the X variables in Figure 9 will be

$$\begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_q \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1_\eta} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2_\eta} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_{q_1} & \lambda_{q_2} & \cdots & \lambda_{q_\eta} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_\eta \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_q \end{bmatrix}$$

However, the measurement model for $\underline{Y}$ is a supermatrix ($\underline{\Lambda}_y$ will be block diagonal),

$$\begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} = \begin{bmatrix} \underline{\Lambda}_{y_{11}} & 0 \\ 0 & \underline{\Lambda}_{y_{22}} \end{bmatrix} \begin{bmatrix} \underline{\eta}_1 \\ \underline{\eta}_2 \end{bmatrix} + \begin{bmatrix} \underline{\varepsilon}_1 \\ \underline{\varepsilon}_2 \end{bmatrix}$$

and the structural model has the form

$$\begin{bmatrix} \underline{I} & \underline{0} \\ -\underline{\beta}_{21} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{\eta}_1 \\ \underline{\eta}_2 \end{bmatrix} = \begin{bmatrix} \underline{\Gamma}_{11} \\ \underline{\Gamma}_{21} \end{bmatrix} \begin{bmatrix} \\ \underline{\xi}_1 \end{bmatrix} + \begin{bmatrix} \underline{\zeta}_1 \\ \underline{\zeta}_2 \end{bmatrix}$$

In the above model $\underline{\xi}_1$ is considered to be a set of background variables which remain constant across time and are included as covariates in the longitudinal model. For pedagogical convenience, let's assume $\underline{\theta}_\varepsilon$, $\underline{\theta}_\delta$, $\underline{\psi}$, and ψ_2 are block diagonals. This means we assume there may be within-occasion but no between-occasion correlation between measurement errors and residuals.

To illustrate the use of path analysis in this model, let's say we wish to determine the components of the variance-covariance matrix $\Omega_{\eta_2\eta_2}$ of the second wave of the endogeneous variables $\underline{\eta}_2$. Figure 10 shows that $\Omega_{\eta_2\eta_2}$ has six components, each of which may be informative to the researcher. These are that the variance-covariance in η_2 is due to:

1. Direct effects from η, not attributable to $\underline{\xi}$, i.e., $\underline{\beta}_{21}\,\underline{\psi}_1\,\underline{\beta}_{21}'$;
2. Indirect effects from $\underline{\xi}_1$ via $\underline{\eta}_1$, i.e., $\underline{\beta}_{21}(\underline{\Gamma}_{11}\underline{\Phi}_{11}\underline{\Gamma}_{11}')\underline{\beta}_{21}'$;
3. Direct effects from $\underline{\xi}_1$, i.e., $\underline{\Gamma}_{21}\underline{\Phi}_{11}\underline{\Gamma}_{21}'$;
4. Two times various indirect effects from $\underline{\xi}_1$ and $\underline{\eta}_1$, i.e., $\underline{\beta}_{21}\underline{\Gamma}_{11}\underline{\Phi}_{11}\underline{\Gamma}_{21}' + \underline{\Gamma}_{21}\underline{\Phi}_{11}\underline{\Gamma}_{11}'\underline{\beta}_{21}'$; and
5. Residual in $\underline{\eta}_2$, i.e., $\underline{\psi}_2$.

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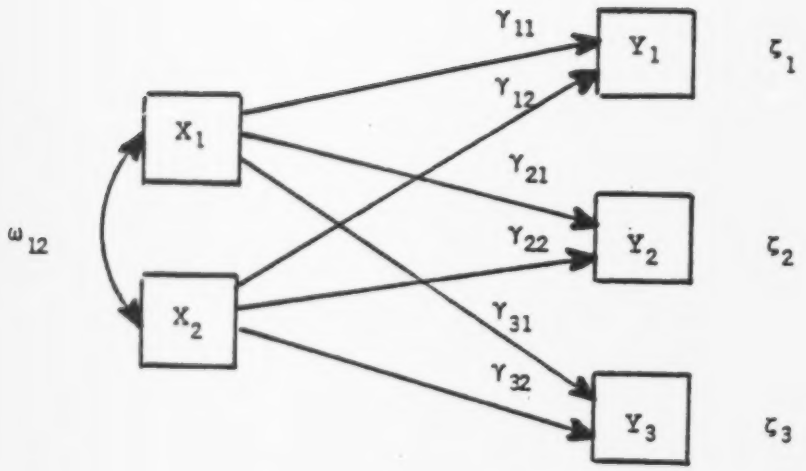


Figure 1. Univariate multiple regression model with three dependent variables regressed on two independent variables.

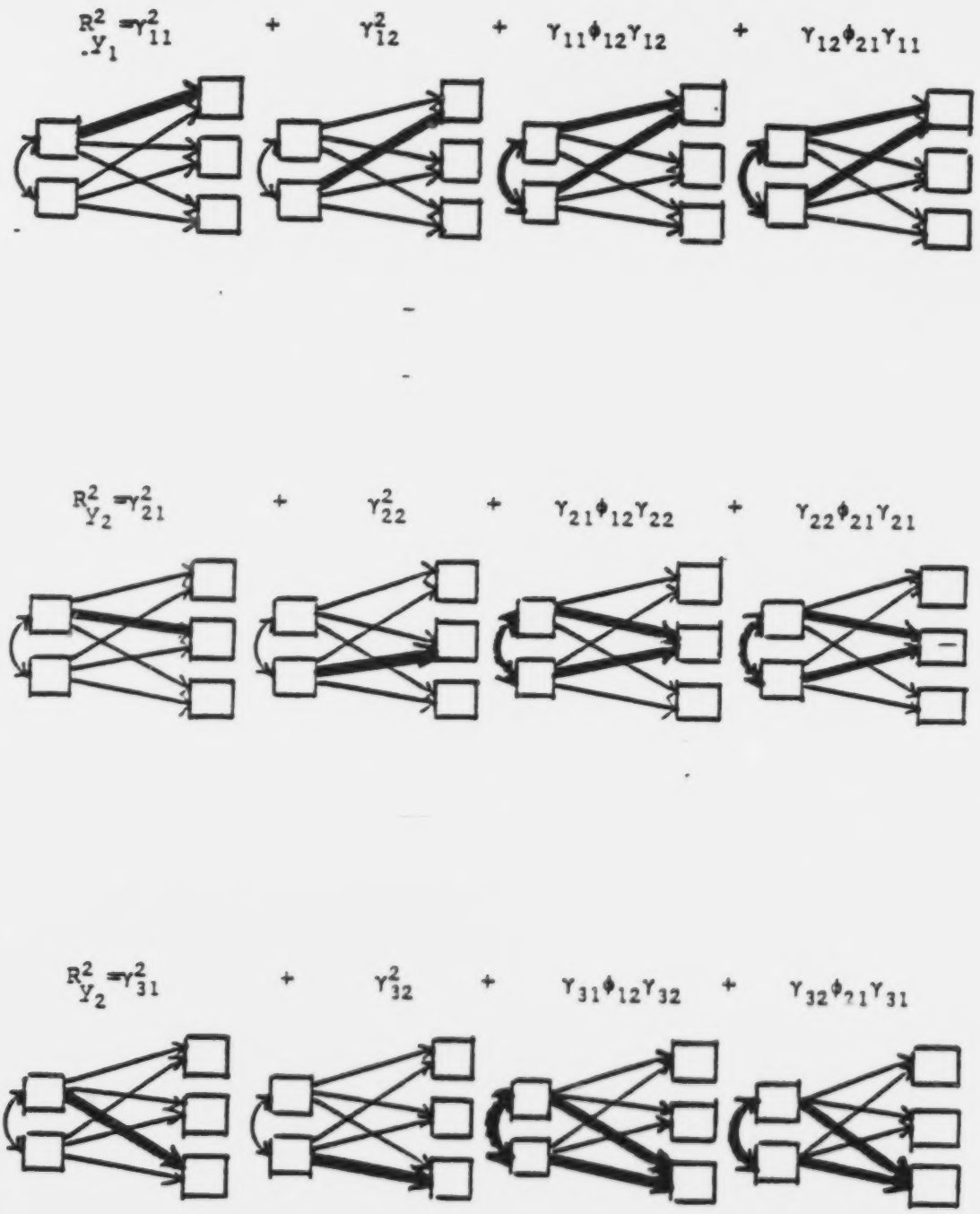

$R^2_{y_1} = \gamma^2_{11} + \gamma^2_{12} + \gamma_{11}\phi_{12}\gamma_{12} + \gamma_{12}\phi_{21}\gamma_{11}$
$R^2_{y_2} = \gamma^2_{21} + \gamma^2_{22} + \gamma_{21}\phi_{12}\gamma_{22} + \gamma_{22}\phi_{21}\gamma_{21}$
$R^2_{y_2} = \gamma^2_{31} + \gamma^2_{32} + \gamma_{31}\phi_{12}\gamma_{32} + \gamma_{32}\phi_{21}\gamma_{31}$

Figure 2. Predicted variance (i.e., R^2) in each dependent variable of the multiple regression model

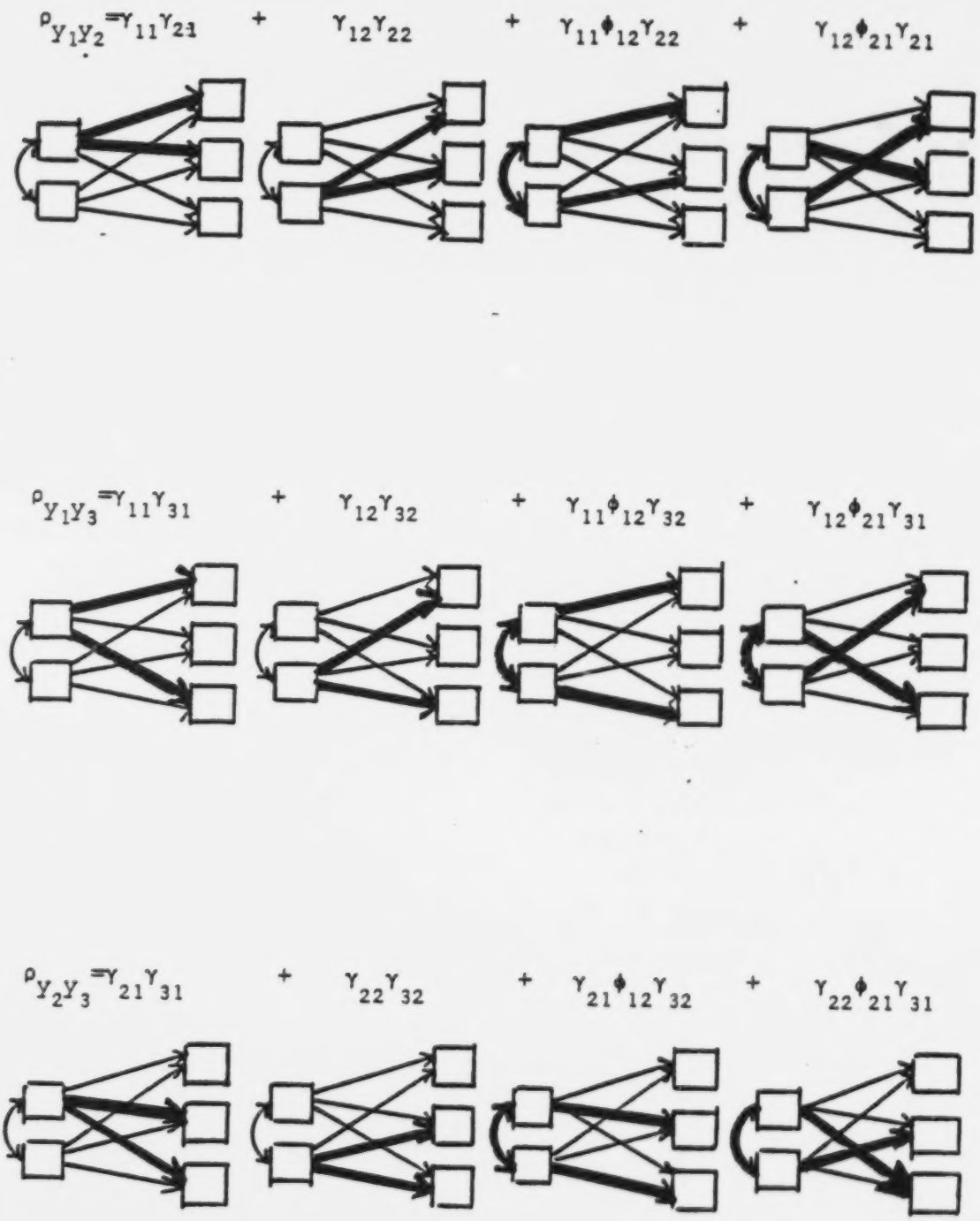

$\rho_{Y_1Y_2} = \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} + \gamma_{11}\phi_{12}\gamma_{22} + \gamma_{12}\phi_{21}\gamma_{21}$
$\rho_{Y_1Y_3} = \gamma_{11}\gamma_{31} + \gamma_{12}\gamma_{32} + \gamma_{11}\phi_{12}\gamma_{32} + \gamma_{12}\phi_{21}\gamma_{31}$
$\rho_{Y_2Y_3} = \gamma_{21}\gamma_{31} + \gamma_{22}\gamma_{32} + \gamma_{21}\phi_{12}\gamma_{32} + \gamma_{22}\phi_{21}\gamma_{31}$

Figure 3. Upper diagonal covariances between each dependent variable for the multiple regression model

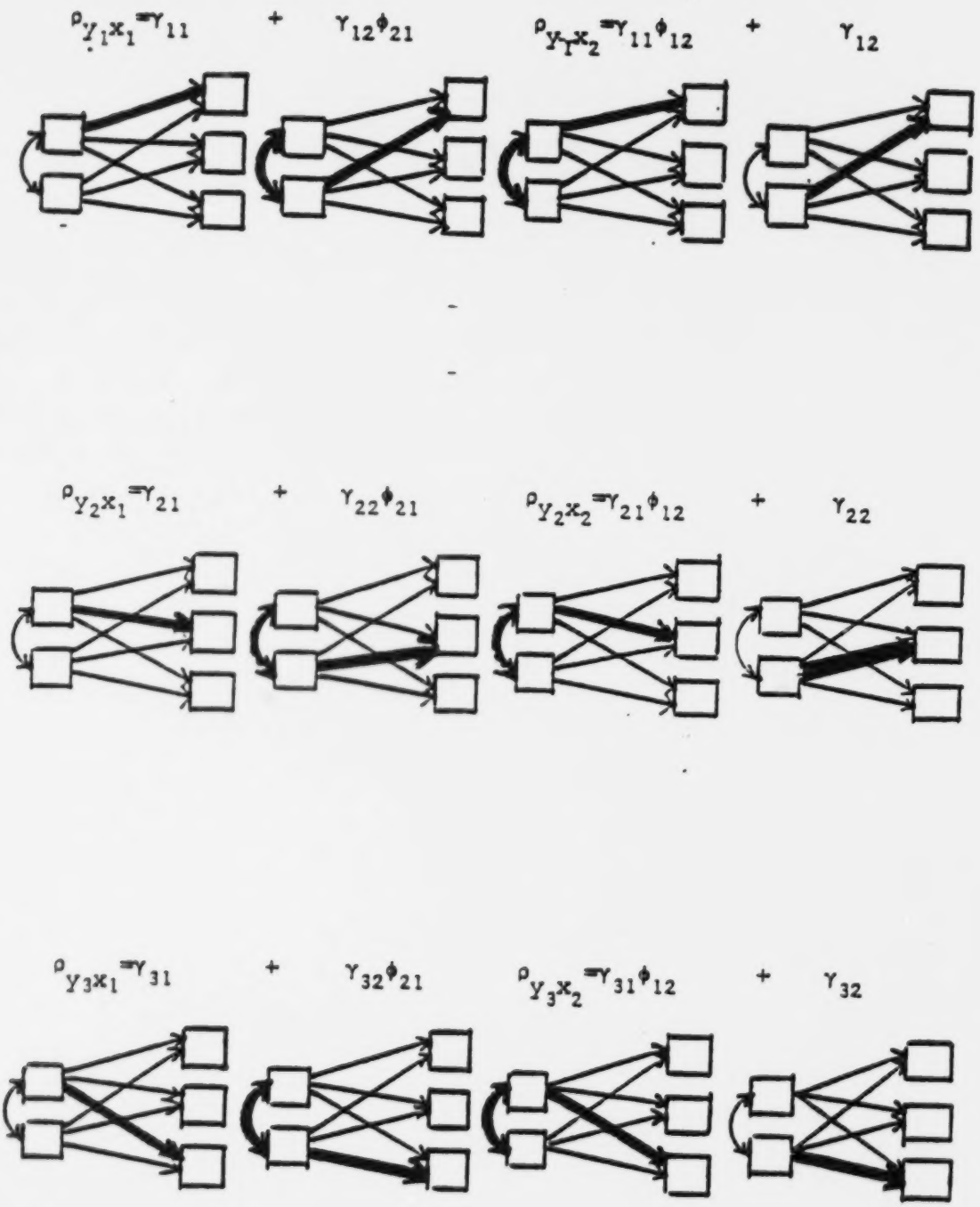

$\rho_{y_1x_1}=\gamma_{11}$ + $\gamma_{12}\phi_{21}$ $\rho_{y_1x_2}=\gamma_{11}\phi_{12}$ + γ_{12}
$\rho_{y_2x_1}=\gamma_{21}$ + $\gamma_{22}\phi_{21}$ $\rho_{y_2x_2}=\gamma_{21}\phi_{12}$ + γ_{22}
$\rho_{y_3x_1}=\gamma_{31}$ + $\gamma_{32}\phi_{21}$ $\rho_{y_3x_2}=\gamma_{31}\phi_{12}$ + γ_{32}

Figure 4. Covariances between each independent and each dependent variable in the multiple regression model

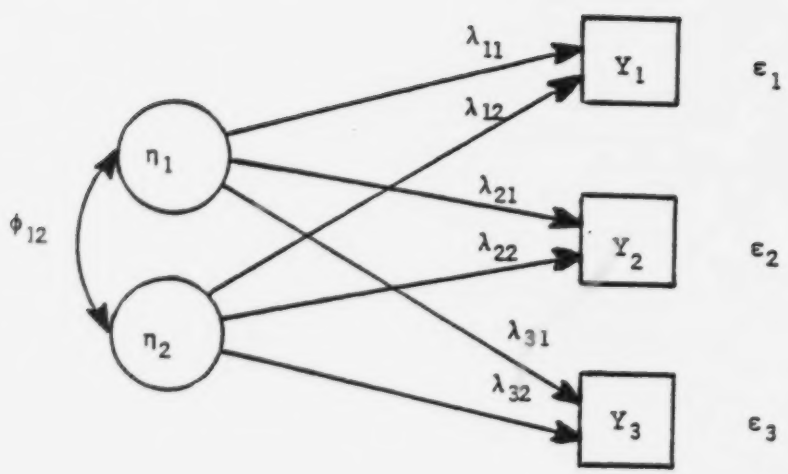


Figure 5. Factor analysis model with three observed variables and two factors.

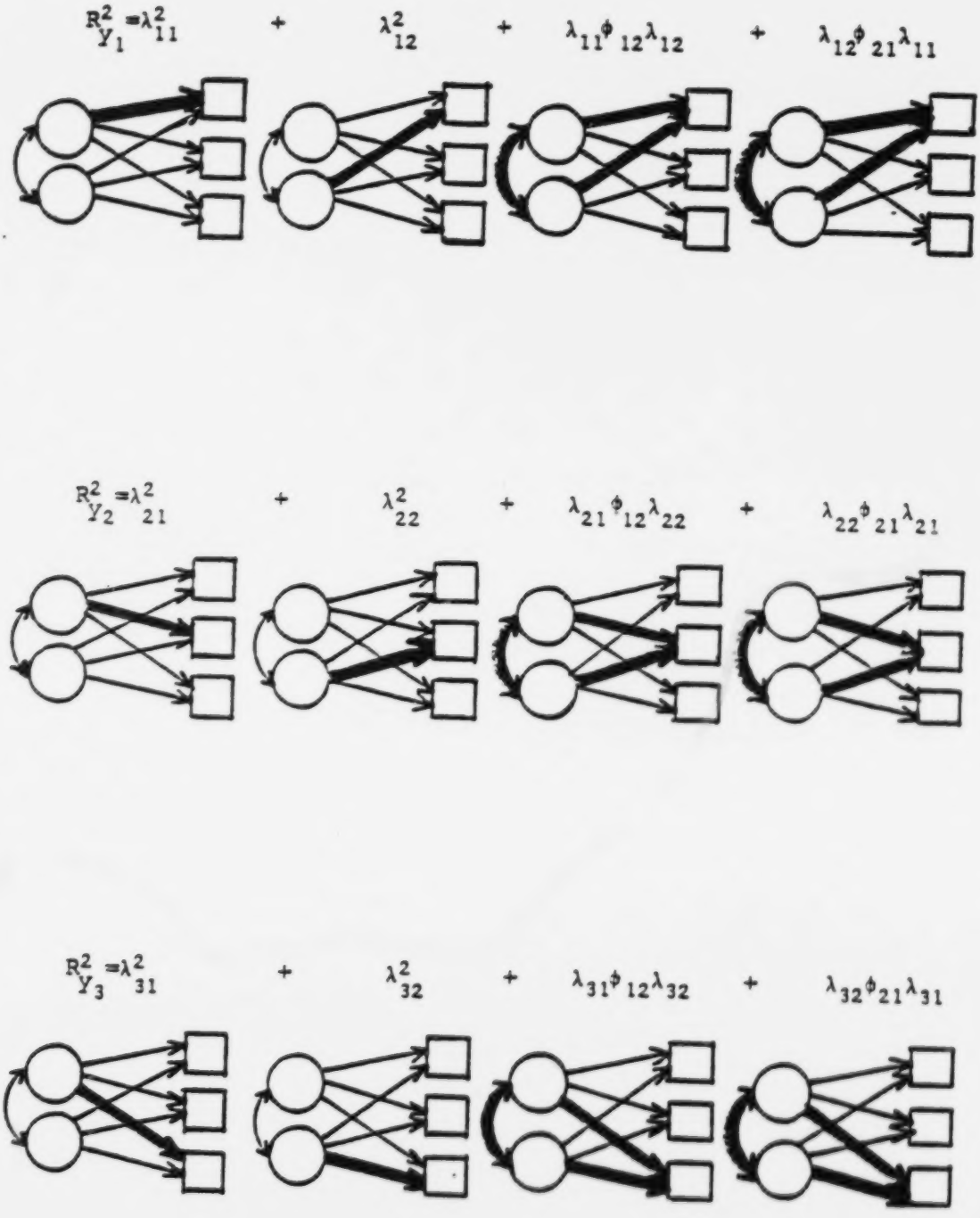
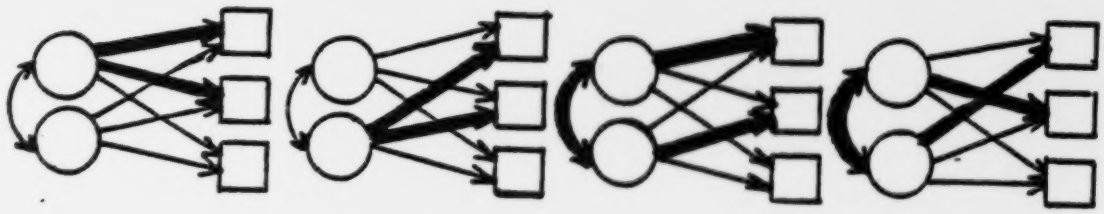
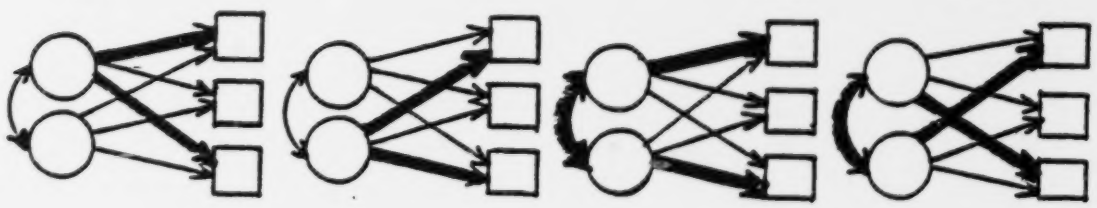

$R^2_{y_1} = \lambda^2_{11} + \lambda^2_{12} + \lambda_{11}\phi_{12}\lambda_{12} + \lambda_{12}\phi_{21}\lambda_{11}$
$R^2_{y_2} = \lambda^2_{21} + \lambda^2_{22} + \lambda_{21}\phi_{12}\lambda_{22} + \lambda_{22}\phi_{21}\lambda_{21}$
$R^2_{y_3} = \lambda^2_{31} + \lambda^2_{32} + \lambda_{31}\phi_{12}\lambda_{32} + \lambda_{32}\phi_{21}\lambda_{31}$

Figure 6. Predicted variances (i.e., communalities) in each observed variable of the factor analysis model

$$\rho_{y_1y_2} = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{11}\phi_{12}\lambda_{22} + \lambda_{12}\phi_{21}\lambda_{21}$$



$$\rho_{y_1y_3} = \lambda_{11}\lambda_{13} + \lambda_{12}\lambda_{32} + \lambda_{11}\phi_{12}\lambda_{32} + \lambda_{12}\phi_{21}\lambda_{31}$$



$$\rho_{y_2y_3} = \lambda_{21}\lambda_{31} + \lambda_{22}\lambda_{32} + \lambda_{21}\phi_{12}\lambda_{32} + \lambda_{22}\phi_{21}\lambda_{31}$$

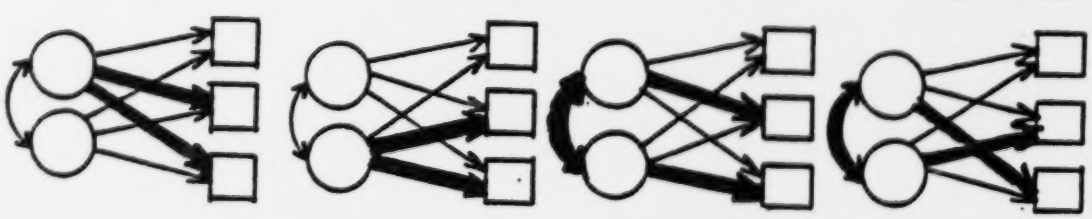


Figure 7. Upper diagonal covariances between each observed variable of the factor analysis model

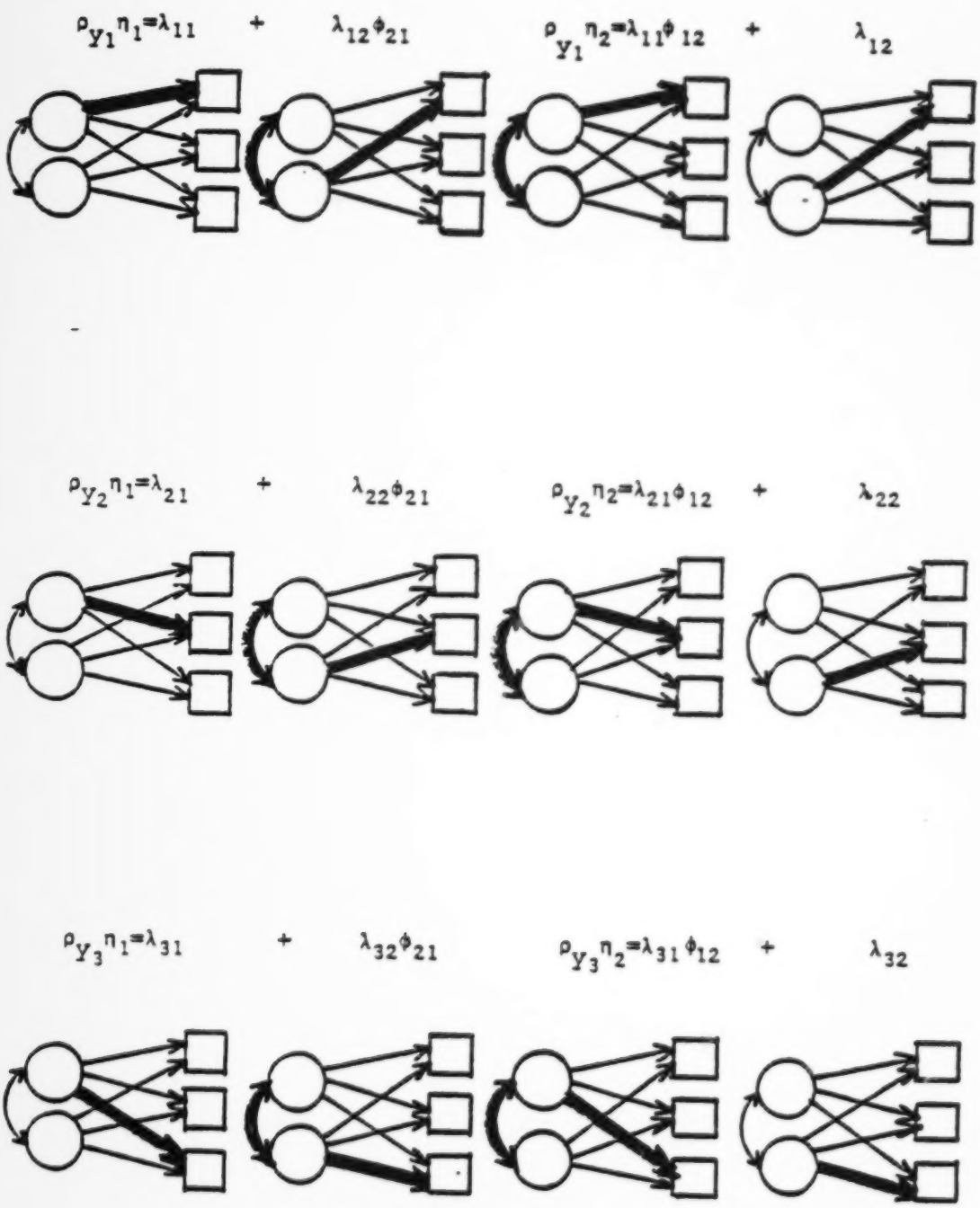

$\rho_{y_1\eta_1}=\lambda_{11}$ + $\lambda_{12}\phi_{21}$ $\rho_{y_1\eta_2}=\lambda_{11}\phi_{12}$ + λ_{12}
$\rho_{y_2\eta_1}=\lambda_{21}$ + $\lambda_{22}\phi_{21}$ $\rho_{y_2\eta_2}=\lambda_{21}\phi_{12}$ + λ_{22}
$\rho_{y_3\eta_1}=\lambda_{31}$ + $\lambda_{32}\phi_{21}$ $\rho_{y_3\eta_2}=\lambda_{31}\phi_{12}$ + λ_{32}

Figure 8. Covariances between each factor and each observed variable (i.e., the factor structure matrix) of the factor analysis model.

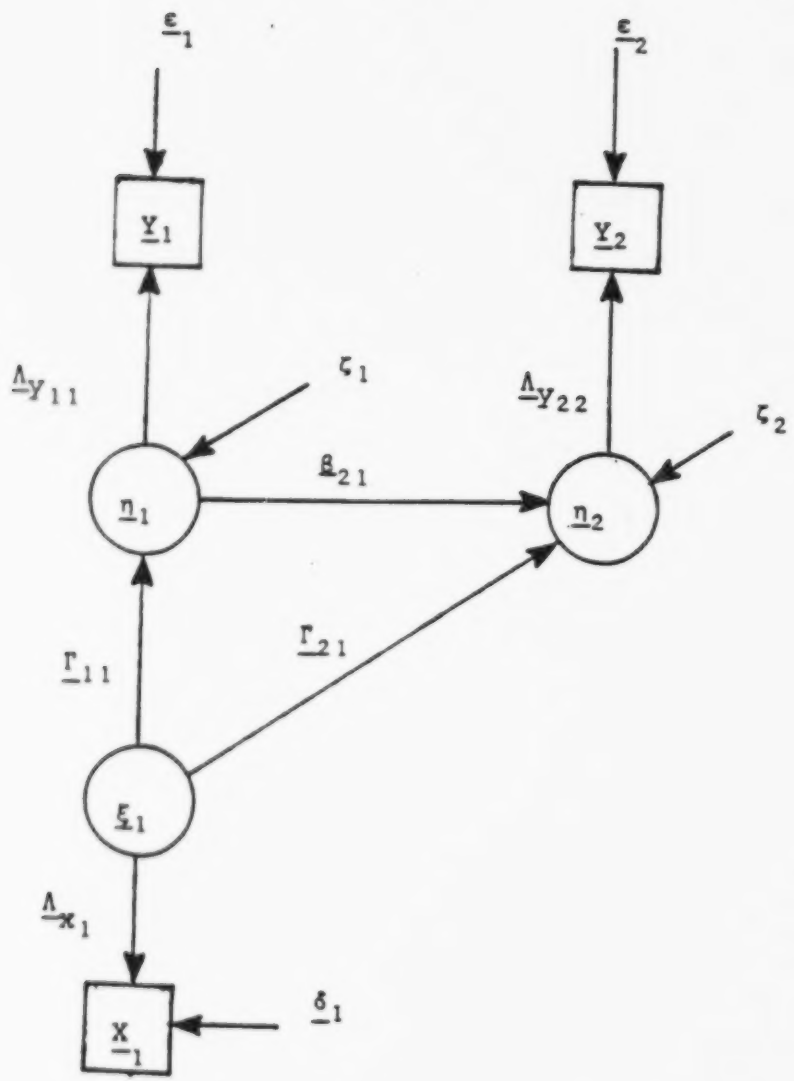


Figure 9. Two-wave multivariable multiple-indicator longitudinal LISREL model

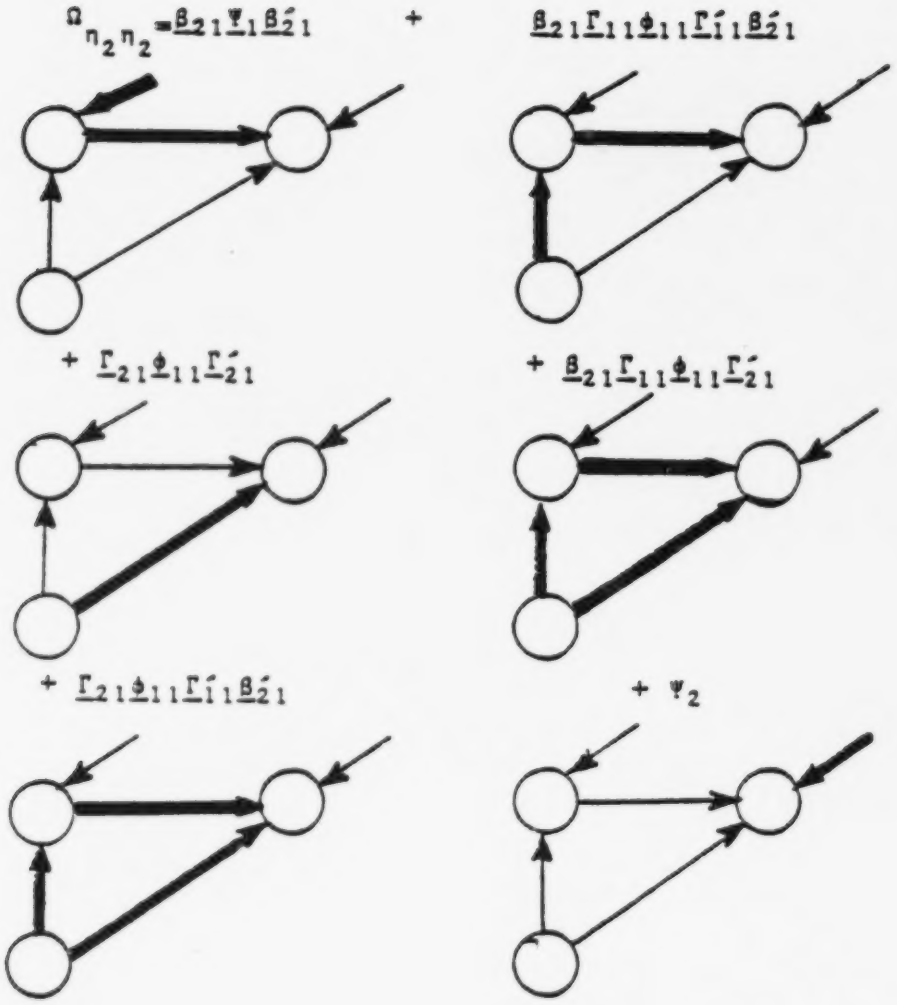


Figure 10. Components of the variance-covariance matrix of the second wave of endogeneous variables